Discharge Algorithms

Michael Durand
Ohio State University
School of Earth Sciences

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Discharge algorithms

- Discharge uncertainty due to SWOT heights?
- Discharge uncertainty due to SWOT temporal sampling?
- Discharge uncertainty due to depth estimates?
- Preliminary SWOT discharge error budget
Sensitivity to SWOT height errors

Use 74 gages with simultaneous height and discharge measurements from North America (USGS), South America (ANA & HyBAm), Bangladesh (IWM)

Biancamaria et al., JSTARS, 2010
Rating curve approach

Power law map from height to discharge

\[ Q = c \left( H_{SWOT} - H_0 \right)^b \]

Power law fit error due to lack of width, slope, backwater conditions:

\[ \eta = \frac{\varepsilon Q}{Q} \]

Biancamaria et al., JSTARS, 2010
Mapping discharge errors

\[
\left( \frac{\sigma Q}{Q} \right)^2 = \eta^2 + \left( b \frac{\sigma_{HSWOT}}{H_{SWOT}} \right)^2
\]

Results vary with river depth

For rivers deeper than about one meter, uncertainty is less than 30%

Rivers deeper than \(~1.5\) m, uncertainty less than 25%

Biancamaria et al., JSTARS, 2010
Discharge algorithms

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Error due to temporal sampling

216 daily streamflow gages sampled with SWOT orbits

Biancamaria et al., JSTARS, 2010
Error varies with drainage area

Smaller rivers are generally “flashier”

Biancamaria et al., JSTARS, 2010
Sampling varies with latitude

Northern rivers sampled more frequently

Biancamaria et al., JSTARS, 2010
Discharge algorithms

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SWOT and river depth

SWOT observes river height down to baseflow depth
Algorithm to estimate river depth

Given SWOT observables: \( T_{s,t} \), \( \frac{\partial h}{\partial x} \bigg|_{s,t} \), \( \Delta y_{s,t} \)

Find the depth estimate at initial time: \( y_{s,1} \)

Assume rectangular channel

Assume uniform flow: \( Q_{s,t} = \frac{1}{n} T_{s,t} \left( \frac{\partial h}{\partial x} \bigg|_{s,t} \right)^\frac{1}{2} (y_{s,1} + \Delta y_{s,t})^{\frac{5}{3}} \)

Pair two pixels, and assume steady flow: \( Q_{s_1,t} = Q_{s_2,t} \)

Rewrite for unknowns: \( \beta_{s_1,t} (y_{s_1,1} + \Delta y_{s_1,t}) = \beta_{s_2,t} (y_{s_2,1} + \Delta y_{s_2,t}) \)

Solve over-constrained problem for unknown depth:

\[ Bx = c \]
\[ x \in \mathbb{R}^2 \]
\[ c \in \mathbb{R}^{nt} \]

Durand et al, JSTARS, 2010
Algorithm testing: Ohio

Durand et al, JSTARS, 2010
Errors in depth and discharge

Depth errors: 18% std.

Includes errors due to SWOT height, baseflow depth, and temporal sampling

Durand et al, JSTARS, 2010
Discharge algorithms

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Manning’s algorithm

\[ Q = \frac{1}{n} w \left( z_0 + \delta z \right)^{5/3} \left( \frac{\partial h}{\partial x} \right)^{1/2} \]

\( w \equiv \text{SWOT observed river width} \)

\( \frac{\partial h}{\partial x} \equiv \text{SWOT observed river slope} \)

\( z_0 \equiv \text{River depth at initial time} \)

\( \delta z \equiv \text{SWOT observed depth changes in time} \)

\( n \equiv \text{Roughness coefficient} \)
Method

- Obtain a set of in situ measurements of width, slope, depth, and roughness estimates
- Obtain uncertainty distributions on each parameter
- Monte Carlo estimates of resulting discharge error distributions
Preliminary Results

Including error on height, slope, bathymetry, roughness coefficient

\[ \frac{\sigma_Q}{Q}, \% \]

Liz Clark et al., in prep.
Preliminary Results

Including error on height, slope, bathymetry, roughness coefficient

$\frac{\sigma_Q}{Q}, \%$

Liz Clark et al., in prep.
Next steps

• A new algorithm:

\[
\overline{Q}_{t_1s_1} - \overline{Q}_{t_2s_2} = \frac{L}{T} \left( \frac{h_{t_2} - h_{t_1}}{\tau_1} \right)^{1/2} \\
\overline{Q}_{t_1s_1} + \overline{Q}_{t_2s_2} = 2K_r \left( \frac{\partial h}{\partial x} \right)
\]

• New approaches to depth estimation (with Larry Smith and Kostas Andreadis)

• Verify all algorithms with AirSWOT
Questions?
Depth results: Cumberland river

Durand et al, JSTARS, 2010
Discharge and river cross sections

**Along-stream (Reach)**

- Water surface
- River Bed

**Across-stream (Cross-section $s_2$)**

- $T_{s_2,t_1}$
- $y_{s_2,t_1}$, $y_{s_2,t_2}$

**Variables**

- $T$: Top width
- $z_{s_1}$, $z_{s_2}$: Bed elevation at cross-sections
- $Q$: River discharge
- $h = z + y$: Water elevation
- $A_{s,t}$: Flow area
- $V = Q/A$: Flow velocity
- $\frac{\partial h}{\partial x} |_t$: Water slope
- $\frac{\partial h}{\partial t} |_s$: River storage change

**Equations**

- $h = z + y$
- $V = Q/A$
Steady algorithm results

Durand et al, JSTARS, 2010
Amazon River discharge estimate

Discharge error less than 10%

Le Favour & Alsdorf, GRL, 2005

- Slope: SRTM heights
- Width: GRFM class.
- Depth: Brazilian nav. charts
- Roughness: Textbook lookup
Algorithm to estimate river depth
Rethinking the flow laws

\[
\frac{\partial Q}{\partial x} + T \frac{\partial h}{\partial t} = 0
\]

\[
\frac{\partial y}{\partial x} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{1}{g} \frac{\partial V}{\partial t} = S_0 - S_f
\]

\[S_f = \frac{Q^2}{K^2}\]

\[K = \frac{1}{n} A^{5/3} P^{2/3}\]

\[Q = AV\]

**Unknowns:**

\[Q, y, A, V, n, P, K, S_0\]
Rethinking a discharge algorithm

1. Apply momentum conservation to a reach, not at a point: Use spatial averages
2. Relax steady flow assumption: Use temporal averages, estimate discharge from river storage change
3. Relax rectangular channel assumption
Relaxing rectangular assumption

A river cross-section

Observations and derivatives

\[ \frac{\partial h}{\partial x} \quad \frac{\partial h}{\partial t} \quad T \quad \delta A \]

Unknown:

\[ A_{base} \]
Assumptions for a new algorithm

\[ \frac{\partial y}{\partial x} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{1}{g} \frac{\partial V}{\partial t} = S_0 - S_f \]

Diffusion Wave Analogy

Apply for a reach without lateral inflows

Wetted Perimeter

\[ P \approx T \]

Stationary Channel

\[ \frac{\partial z}{\partial t} = 0 \quad \frac{\partial n}{\partial t} = 0 \]
Conservation of mass

$$\frac{\partial Q}{\partial x} + T \frac{\partial h}{\partial t} = 0$$

Apply to a reach for period $\tau_1$ between overpasses $t_1$, $t_2$

$$\int_{s_1}^{s_2} \int_{t_1}^{t_2} \left( \frac{\partial Q}{\partial x} + T \frac{\partial h}{\partial t} \right) ds dt = 0$$

Average flow for $\tau_1 \equiv Q_{s, \tau_1} = \frac{1}{\tau_1} \int_{t_1}^{t_2} Q_s(t) dt$
Conservation of mass

Define similar averages for top width and height

\[ \bar{h}_t = \frac{1}{L} \int_{s_1}^{s_2} h_t(x) \, dx \]

\[ L \equiv \text{reach length} \]

Mass conservation becomes:

\[ \bar{Q}_{s_1} - \bar{Q}_{s_2} = \frac{L}{2\tau_1} \left( \bar{T}_t + \bar{T}_{t+1} \right) \left( \bar{h}_{t_2} - \bar{h}_{t_1} \right) \]

This equation is exact for the average quantities
Momentum equation for a reach:

\[
\frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} - S_0
\]

\[
\frac{\partial h}{\partial x} = S_f = \frac{Q^2}{K^2}
\]

Apply between two overpasses for one reach:

\[
\overline{Q}_{s_1} + \overline{Q}_{s_2} = 2K \left( \frac{\partial h}{\partial x} \right)^{1/2}
\]
Unsteady algorithm: Conveyance

Apply to a reach $r_{12}$ from $s_1$, $s_2$ averaged from $t_1$, $t_2$

\[
\overline{Q}_{s_1} + \overline{Q}_{s_2} = 2\overline{K}_{r_{12}} \left( \frac{\partial h}{\partial x} \right)^{1/2}
\]

\[
\overline{K}_r = \frac{K_{r,t_1} + K_{r,t_2}}{2}
\]

\[
K_{r,t} = \frac{1}{n_r} \left( A_{base,r} + \delta A_{r,t} \right)^{5/3} T_{r,t}^{2/3}
\]

Two time-invariant unknowns per reach
Application to a single reach

At $t_2$ two equations, four unknowns

$$\overline{Q}_{t_1t_2s_1} - \overline{Q}_{t_1t_2s_2} = LT \left( \frac{h_{t_2} - h_{t_1}}{\tau_1} \right)^{1/2}$$

$$\overline{Q}_{t_1t_2s_1} + \overline{Q}_{t_1t_2s_2} = 2\overline{K}_r \left( \frac{\partial h}{\partial x} \right)^{1/2}$$

At $t_3$ four equations, six unknowns

$$\overline{Q}_{t_3t_2s_1} - \overline{Q}_{t_3t_2s_2} = LT \left( \frac{h_{t_3} - h_{t_2}}{\tau_2} \right)^{1/2}$$

$$\overline{Q}_{t_3t_2s_1} + \overline{Q}_{t_3t_2s_2} = 2\overline{K}_r \left( \frac{\partial h}{\partial x} \right)^{1/2}$$

At $t_3$ six equations, eight unknowns...
Application to two reaches

At $t_2$:

$$Q_{t_{12}s_1} - Q_{t_{12}s_2} = L_1 \tau_1^{-1} T_{r_1} (h_{t_2} - h_{t_1})$$

$$Q_{t_{12}s_2} - Q_{t_{12}s_3} = L_2 \tau_1^{-1} T_{r_2} (h_{t_2} - h_{t_1})$$

$$Q_{t_{12},s_1} + Q_{t_{12},s_2} = 2K_{r_{12}} \left( \frac{\partial h}{\partial x} \bigg|_{r_{12}} \right)^{1/2}$$

$$Q_{t_{12},s_2} + Q_{t_{12},s_3} = 2K_{r_{23}} \left( \frac{\partial h}{\partial x} \bigg|_{r_{12}} \right)^{1/2}$$

Write two equations per reach, four total

We have three discharge unknowns and two time-invariant conveyance unknowns per reach: seven total
At $t_3$ add these equations:

\[
\begin{align*}
\overline{Q}_{t_{23}s_1} - \overline{Q}_{t_{23}s_2} &= L_1 \tau_2^{-1} T_{r_1} (h_{t_3} - h_{t_2}) \\
\overline{Q}_{t_{23}s_2} - \overline{Q}_{t_{23}s_3} &= L_2 \tau_1^{-1} T_{r_2} (h_{t_3} - h_{t_2}) \\
\overline{Q}_{t_{23},s_1} + \overline{Q}_{t_{23},s_2} &= 2K_{r_{12}} \left( \frac{\partial h}{\partial x} \right)_{r_{12}}^{1/2} \\
\overline{Q}_{t_{23},s_2} + \overline{Q}_{t_{23},s_3} &= 2K_{r_{23}} \left( \frac{\partial h}{\partial x} \right)_{r_{12}}^{1/2}
\end{align*}
\]

We now have eight equations.

We have six discharge unknowns but the same two time-invariant conveyance unknowns per reach: ten total.
Application to two reaches

At \( t_4 \) we have twelve equations.

We have nine discharge unknowns but the same two time-invariant conveyance unknowns per reach: thirteen total.

At \( t_5 \) we have sixteen equations, and sixteen unknowns.

At \( t_6 \) we have twenty equations, and nineteen unknowns.

Six overpasses required.
For three or more reaches, only four overpasses are needed for equations to outnumber unknowns.
Checking diffusive assumption

After solution is obtained, calculate \( V = Q/A \)

Linear momentum equation:

\[
\frac{\partial y}{\partial x} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{1}{g} \frac{\partial V}{\partial t} = S_0 - S_f
\]

Calculate magnitude of momentum terms, verify assumption:

\[
\frac{V}{g} \frac{\partial V}{\partial x} + \frac{1}{g} \frac{\partial V}{\partial t}
\]
Next steps

- Build a hydraulic framework capable of testing the algorithm
- Examine assumptions, determine in what cases it would fail

\[
\frac{\partial Q}{\partial x} + T \frac{\partial h}{\partial t} = 0
\]

\[
\frac{\partial y}{\partial x} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{1}{g} \frac{\partial V}{\partial t} = S_0 - S_f
\]